Int. J. Solids Structures Vol. 27, No. 12, pp. 1551–1573, 1991
Printed in Great Britain.

0020-7683/91 $3.00 + .00

BIFURCATION HIERARCHY OF SYMMETRIC STRUCTURES

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(*Received* 24 *November* 1989; *in revised form* 3 *July* 1990)

Abstract—A group-theoretic method for the analysis of bifurcation behavior of regular-polygonal symmetric structures is described. Possible bifurcation paths and points of these structures are categorized in terms of dihedral and cyclic groups, which express the symmetry. In particular, we offer a complete description of those double bifurcation points which occur due to group symmetry. The type, the number, and the stability of bifurcation paths branching at these points are determined by deriving bifurcation equations. The existence of a potential function plays a substantial role for the existence of bifurcation paths. As a result of these, all possible bifurcation process can be known *a priori* as a natural consequence of bifurcation hierarchy before actual numerical analysis. An implementation of this method in numerical analysis shows its validity and usability.

1. INTRODUCTION

Structures with symmetry have played an important role in the history of human architectures from the Greek or Roman eras. Among modern architectures, we may also find structures which preserve some symmetry. The reticulated truss domes of Fig. 1 may serve as such an example.

Such symmetry, however, can often be broken at the onset of bifurcation buckling behavior, which is one of the typical collapse types of structures. Naturally, theoretical study and computer-assisted numerical analysis of this behavior are of most practical importance.

Such buckling behavior can be interpreted as an instability induced by a singular tangent-stiffness matrix—linearized eigenvalue problem—of structural systems. Singular (critical) points are where one or more eigenvalues of this matrix vanish, and it is at such points (with some additional conditions) where bifurcation buckling actually takes place. In fact, from a singular point on a fundamental solution path, secondary (post-buckling) paths bifurcate, the number of which being in general greater than one.

In view of its practical and theoretical importance, the problem of elastic instability has been the subject of a number of extensive theoretical studies. For instance, the static perturbation method is an established means of studying elastic buckling, imperfection sensitivities, and so on; see, e.g., Hutchinson and Koiter (1970) and Thompson and Hunt (1973). The eigenvalue-analysis method seems to be an alternative in computer-assisted numerical buckling analysis; see, for example, Timoshenko and Gere (1961).

It should be noted that, at the expense of beauty, symmetric structures have much more complex bifurcation behavior. By "bifurcation behavior", we mean the number, the stability, and the symmetry of bifurcating branches. More precisely, multiple critical points, where more than one eigenvalue simultaneously vanishes, appear inherently in such structures due to the presence of symmetry. The essential point here is that such multiplicity appears in a generic manner. This "genericity" implies that such singularities appear with a change of a *single* physical parameter (e.g. the loading parameter).

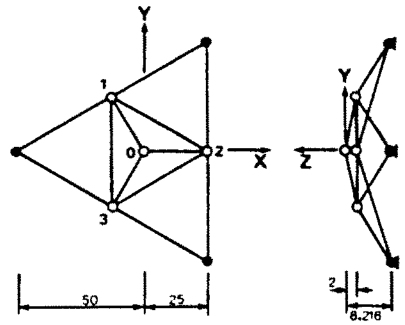

Fig. 1(a). Regular-triangular truss dome (D_3-equivariant).

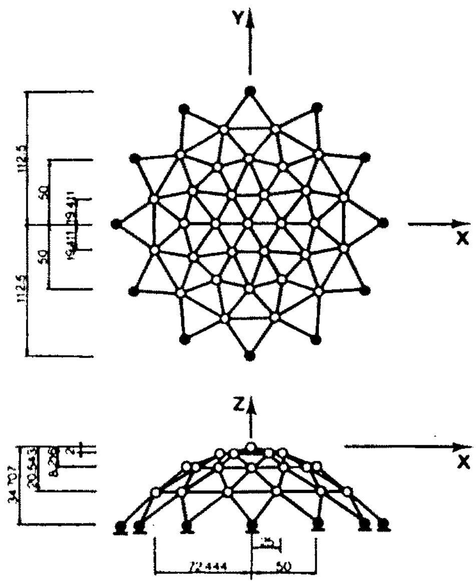

Fig. 1(b). Spherical diamond shell (D_6-equivariant).

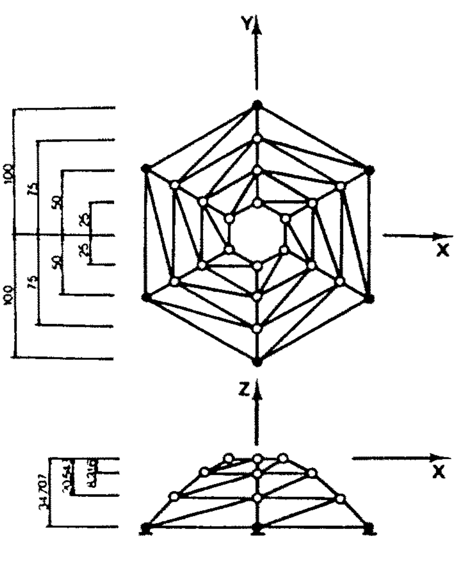

Fig. 1(c). Schwedler dome (C_6-equivariant).

We classify multiple critical points into two types: parametric and group-theoretic. Group-theoretic multiple points are those which appear inherently in structures even with one physical parameter if they have some symmetry. On the other hand, parametric multiple points are those which appear as a coincidence of a pair of simple critical points. In physical experiments they are observed only when more than one physical parameter is changed, and hence are rare (i.e. non-generic) in customary structural analysis.

Accordingly, the critical points which appear in a symmetric structure consist of two major types, simple critical points and multiple ones due to symmetry. This shows a sharp contrast with the case of non-symmetric structures, where only simple critical points appear generically. Buckling and bifurcation at simple points have been extensively studied (see, e.g., Thompson and Hunt, 1973; Sattinger, 1979, 1980; Fujii and Yamaguti, 1980; Hunt, 1986). However, the importance of group-theoretic multiple points can never be ignored in the case of symmetric structures.

With regard to the static perturbation method, the degeneracy (multiplicity) due to the group-theoretic symmetry results in annihilating some lower-order derivatives of the total potential energy function. We should note that this degeneracy has so far been treated in an *ad hoc* manner within this method.

The alternative method of bifurcation analysis, i.e. the eigenvalue analysis, finds a bifurcation path at a multiple point by means of an iterative procedure starting from a *plausible* branching direction which is selected on a trial-and-error basis. This method is more convenient and less costly than the former if it is adequate. Although the critical eigenvectors at a multiple point form a multi-dimensional space, only a finite number of these correspond to the directions of bifurcation paths, as Hosono (1976) has noted in an empirical manner. The crucial difficulty in this method is that, notwithstanding its merits, it can never guarantee in principle to yield the complete set of bifurcation paths at a multiple bifurcation point. In short, the difficulty in numerical analysis mainly comes from the inherent group-theoretic degeneracy.

Summing up the preceding arguments, vital questions to be answered seem to be:

(Q1) how the *hierarchy* of successive bifurcations is described in terms of the group symmetry of the system?
(Q2) how the *symmetry* of bifurcation paths is determined?
(Q3) how the *number* of bifurcation paths is determined?
(Q4) how the *direction* of bifurcation paths is found?

In view of recent developments of group-theoretic bifurcation theory in the field of non-linear mathematics, bifurcation structures near singular points, whether simple or multiple, can be investigated theoretically. In particular, questions as above can be completely answered *a priori* if the concept of group symmetry is taken into account in the theory. A combination of group-theoretic bifurcation theory with the conventional static perturbation or eigenvalue method appears to give a more comprehensive and simpler way of describing and tracing bifurcation behavior.

The aim of this paper is, through an extension and reorganization of those results hitherto developed on group-theoretic double singularity, to make the bifurcation structure more transparent, and provide the results in a form readily accessible for engineers. Since they include *a priori* information on the structure of bifurcations near a multiple point (as will be shown later), such mathematical results seem to serve as a basic tool for engineers to perform numerical analysis, e.g. by the eigenvalue method. Emphasis is placed on dihedral groups D_n, i.e. groups of regular polygons, which appear in many structures of practical importance.

An additional note, which appears to be not fully recognized so far, is given about a system which is equivariant to a cyclic group C_n (namely, a system with rotational n-gon symmetry, but without reflections); a C_n-equivariant system arises from a structure with cyclic symmetry or from a secondary bifurcated state of a structure with dihedral symmetry. Since many structures are in general *potential systems*, the bifurcation behavior of structural systems shows some special features. For details, see Appendix A.

We shall present a complete diagram of bifurcation hierarchy which can appear as D_n- and C_n-equivariant systems. This diagram gives complete information about possible types of successive bifurcations, the number of bifurcating branches, their stability, and so on.

In fact, referring to this hierarchy diagram, we can understand a very complex bifurcation behavior of symmetric truss domes. The diagram explains the empirical observation (cf. Section 6 and also Ikeda and Torii, 1986) that a C_3-symmetric path can never bifurcate directly from a D_6-symmetric path. It is to be noted that all conclusions drawn here are applicable to bifurcation phenomena of systems other than truss domes, whenever they have some dihedral or cyclic symmetry.

The mathematical tools employed in this paper are:

(1) group-theoretic bifurcation theory (see, e.g., Sattinger, 1979),

(2) application of this theory to D_n (Sattinger, 1979, 1983; Fujii *et al.*, 1982; Golubitsky *et al.*, 1988; Healey, 1985, 1988; Dellnitz and Werner, 1989),

(3) categorization of group-theoretic double points of D_n-equivariant truss dome structures by means of the index (Ikeda and Torii, 1986),

(4) results of standard (isotypic) decomposition that provide us with information about the symmetry of eigenfunction (Fujii *et al.*, 1982), and

(5) a remark on a C_n-equivariant system, and especially an existence proof of bifurcating branches for potential systems (Krasnosel'skii, 1964; Poston and Stewart, 1978).

A few remarks follow. Firstly, concerning "parametric" multiple points, extensive studies have been made for systems of reaction-diffusion equations; see, e.g., Fujii *et al.* (1982) and references therein.

Secondly, the concept of *index*, which denotes the difference of the level of symmetry of the fundamental and bifurcated paths, has already been used as a characteristic of symmetric truss domes by Ikeda and Torii (1986). Such results will be systematically summarized and extended in this paper.

The final remark is about C_n-equivariant systems. For general C_n-symmetric systems, it has been "well" recognized that secondary paths cannot branch from a group-theoretic double point (see, e.g., Sattinger, 1979). However, bifurcation paths do branch at a group-theoretic double point for a C_n-equivariant structural system that usually has a potential function. Although this fact has been known in mathematics (e.g. Krasnosel'skii, 1964), we shall offer a brief account on this in view of its importance in structural systems.

2. A SIMPLE EXAMPLE

Before entering into general settings, we shall provide here a simple example of bifurcation behavior that will serve to give concrete ideas to the arguments in subsequent sections. The objective of this section is to identify the problems to be addressed later and to illustrate the use of group-theoretic terminology.

We consider the elastic regular-triangular truss dome of Fig. 1(a), subjected to a symmetric loading. All the members are assumed to have the same material and sectional properties. This dome is apparently symmetric in geometric configuration, in stiffness distribution and in loading. The equilibrium equations for this dome remain invariant under two kinds of geometric transformations, namely the counterclockwise rotation r around the Z-axis at an angle of $2\pi/3$ and the reflection s with respect to XZ-plane. This geometric invariance is mathematically expressed as the equivariance of the bifurcation problem to a group, i.e. the dihedral group of degree three†

$$D_3 = \{e, r, r^2, s, sr, sr^2\},$$

where the element e stands for the identity transformation, and r^j stands for counterclockwise rotation around the Z-axis at an angle of $2\pi j/3$ ($j = 1, 2$).

† In the Schoenflies notation this group is denoted as C_{3v}, whereas D_3 means another (isomorphic) group in which s represents the half-rotation around the X-axis. They are isomorphic as abstract groups but are distinguished as subgroups of all isometries.

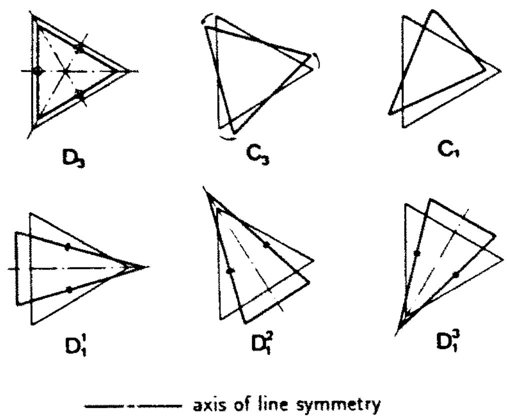

Fig. 2. Plane views of bifurcation patterns of regular-triangular free nodes of the regular-triangular dome.

Deformation (bifurcation) patterns of this dome are often less symmetric than D_3 but may retain part of its symmetry, which is represented by its invariance under the transformations in the subgroups of D_3 i.e.

$$D_3,\quad C_3 = \{e, r, r^2\},\quad D_1^j = \{e, sr^{j-1}\}\ (j = 1, 2, 3),\quad C_1 = \{e\}.$$

Figure 2 shows plane views of deformation patterns of the regular-triangular free nodes 1, 2 and 3 of this dome. Solid–dash lines denote the axis of line symmetry. Group D_3 denotes a uniform expansion or contraction of the regular triangle, accompanied by a uniform float or drop. Group C_3 expresses a rotated-regular-triangular pattern indicating a rotation about the Z-axis, accompanied also by a uniform expansion or contraction and by a uniform float or drop. Group D_1^j ($j = 1, 2, 3$) indicates an isoceles-triangular pattern with one axis of line symmetry. The patterns for D_1^1, D_1^2 and D_1^3 are symmetric with respect to rotations of $2\pi j/3$ ($j = 1, 2$). Group C_1 represents a completely asymmetric scalene-triangular pattern.

The dome displays a highly-complex bifurcation phenomenon, which has been computed by means of a finite-displacement bifurcation analysis technique (see Nishino *et al.*, 1984). Here the external loads are proportional to a constant loading pattern vector and magnified by a loading parameter f; the vertical components of nodes 1, 2 and 3 of this pattern vector are equal to unity and all other components to zero. Figure 3 shows (a) space and (b) plane views of the equilibrium paths, which display different aspects of the same bifurcation phenomenon. The former shows the relationship among the loading parameter f and X- and Y-directional displacements of the center node 0; the latter displays the relationship between f and the vertical displacement of node 0. The symbol (●) denotes a symmetric simple bifurcation point and (△) an asymmetric double. As many as six bifurcation paths branch at the unstable asymmetric double bifurcation points a and b, respectively [see Fig. 3(a) for point a]. From these bifurcation paths, further branches bifurcate at simple bifurcation points c and d.

We may observe that the deformation patterns of the dome on each path remain symmetric with respect to a subgroup of D_3. Although the deformation of the dome continuously progresses with changes in f, all deformation patterns of the dome on the fundamental path (denoted by the dark continuous line) are necessarily D_3-symmetric. Likewise, all the patterns on the paths expressed by the long-dashed lines are D_1^j-symmetric; those on dotted lines C_1-symmetric. Thus, each path is associated with a particular symmetry expressed by a subgroup of D_3. Since D_3, D_1^j and C_1 are more symmetric in this order, one can regard this bifurcation behavior as a process of symmetry breaking [cf. (Q1) in the Introduction]. At a bifurcation point, the path with higher symmetry can be called the main path, whereas the others with less symmetry, the bifurcation paths.

We may also observe that the symmetry of the paths is closely related to the type of bifurcation points [cf. (Q2) in the Introduction]. The asymmetric double points a and b are

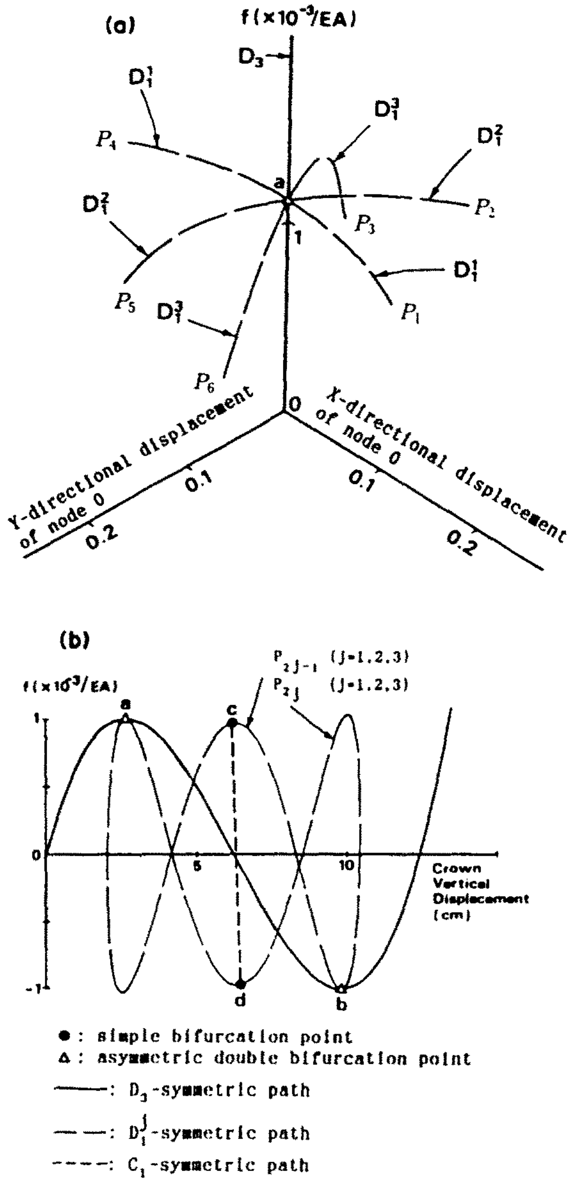

Fig. 3. Equilibrium paths of the regular-triangular dome (D_3-equivariant). (a) space view; (b) plane view.

the points where the D_1^j-symmetric bifurcation paths ($j = 1, 2, 3$) branch from the D_3-symmetric main path. The symmetric simple points c and d are where the C_1-symmetric bifurcation paths ($j = 1, 2, 3$) bifurcate from the D_1^j-symmetric main paths. Group-theoretic considerations will reveal how the symmetry of a bifurcating path is related to the type of the bifurcation point (to be summarized in Table 1 in Section 5).

At the double bifurcation point a on the D_3-symmetric path, we have computed six bifurcation paths $P_1, \ldots, P_6$. How can we convince ourselves that they exhaust all bifurcation paths branching at this point [cf. (Q3) in the Introduction]? Group-theoretic considerations answer this without numerical computations.

The critical eigenvectors at a, forming a two-dimensional subspace, have the symmetry of D_1^j ($j = 1, 2, 3$) or C_1. We observe that all the six bifurcation paths $P_1, \ldots, P_6$ are in the directions with higher symmetry, i.e. with D_1^j-symmetry ($j = 1, 2, 3$). Why are such directions chosen and is this a general phenomenon [cf. (Q4) in the Introduction]? This question will be answered by group-theoretic analysis (see Proposition 6 in Section 4).

The paths P_j and P_{j+3} are D_1^j-symmetric ($j = 1, 2, 3$). Deformation modes for P_1, P_3 and P_5 (respectively P_2, P_4 and P_6) are mutually rotation-symmetric. We only have to obtain paths P_1 and P_4 in the numerical analysis, whereas the others are known through geometric symmetry. Note that by virtue of this symmetry the six paths in the space view (a), degenerate into only two paths in the plane view (b). In the field of structural engineering, it is customary to employ plane views to describe bifurcation behavior, so that the rotation-symmetric paths are identified automatically.

We realize that the apparently complex bifurcation behavior of the dome does not occur randomly but occurs quite systematically: symmetry is the underlying rule which controls the behavior. The characteristics of the bifurcation paths and points will be systematically categorized in the following sections by means of a group-theoretic bifurcation theory, which can describe the symmetry in a relevant manner.

3. GROUP THEORY FOR BIFURCATION BEHAVIOR

The nonlinear equations of a multi-dimensional system read:

$$\underline{H}(f, \underline{x}) = \underline{0}, \tag{1}$$

where f is a real loading (bifurcation) parameter; $\underline{x}$ represents a nodal displacement vector, which is an independent variable belonging to an N-dimensional real space $X = \mathbf{R}^N$; and $\underline{H}$ are equilibrium equations denoting a nonlinear continuous mapping from $\mathbf{R} \times \mathbf{R}^N$ into $\mathbf{R}^N$. We define $\underline{H}$ in such a manner that the eigenvalues of the Jacobian (tangent-stiffness) matrix J of $\underline{H}$ are all positive at $(f, \underline{x}) = (0, \underline{0})$. This means that the system is originally (subcritically) in a stable state.

The geometric symmetry† of the equations $\underline{H}$ is described by a group (for the necessary background in group theory, see Miller, 1972; Serre, 1977; Dinkevich, 1984; Golubitsky *et al.*, 1988). Let G be a finite group, and $T(g)$ for $g \in G$ denote an $N \times N$ orthogonal matrix of a linear representation of G on X. Then group G is defined to be the symmetry group of the equations $\underline{H}$, if $\underline{H}$ are equivariant (covariant) to G in the sense that

$$T(g)\underline{H}(f, \underline{x}) = \underline{H}(f, T(g)\underline{x}), \quad \text{for all } g \in G. \tag{2}$$

Equivariance of equations to a group indicates that they are invariant under the transformation by all elements of the group. This equivariance is inherited to J:

$$T(g)J(f, \underline{x}) = J(f, T(g)\underline{x})T(g), \quad \text{for all } g \in G.$$

In particular, if $\underline{x}$ is invariant under G, i.e. $T(g)\underline{x} = \underline{x}$ for all $g \in G$, then

$$T(g)J(f, \underline{x}) = J(f, \underline{x})T(g), \quad \text{for all } g \in G.$$

The Lyapunov-Schmidt decomposition (Sattinger, 1979; Golubitsky and Schaeffer, 1985; Golubitsky *et al.*, 1988) reduces the original equations to a fewer number of bifurcation equation(s) in the kernel space of J, spanned by the critical eigenvector(s) of J, as

$$\underline{h}(f, \underline{w}) = \underline{0}, \tag{3}$$

where

$$\underline{h} = (h_1, \ldots, h_M)^T$$

and M is the dimension of the kernel space ($M \leqslant N$), $\underline{w}$ is an M-dimensional real independent variable vector, and $(\cdot)^T$ is the transpose of a vector. Since the original equations

† It should be understood that the group G is not necessarily purely "geometric" but is the intersection of the geometric and the material symmetry of the system, as well as the symmetry of parametric load.

H are equivariant to G, the bifurcation equations can be chosen to be equivariant to G, that is,

$$t(g)h(f, w) = h(f, t(g)w), \quad \text{for all } g \in G, \tag{4}$$

where $t(g)$ is an $M \times M$ matrix denoting the subrepresentation of $T(g)$ on $\mathbf{R}^M$.

The type of equilibrium points is to be determined according to the rank deficiency M of J as below:

- ordinary point: if $M = 0$,
- simple critical point: if $M = 1$,
- double critical point: if $M = 2$, and so on.

We further divide the double point into:

- group-theoretic: if the kernel of J is G-irreducible,
- parametric: if it is G-reducible,

where "G-irreducible" means that there exist no non-zero proper G-invariant subspaces of the kernel of J (Golubitsky *et al.*, 1988). In this paper, we shall be mainly interested in group-theoretic double points, which appear generically in structural analysis, and do not deal with parametric double points, which are rare.

In order to describe systems with regular-polygonal symmetry the equations H are hereafter assumed† to be equivariant to the dihedral group D_n of degree n. This group‡, representing the symmetry of a regular n-gon, is defined as

$$D_n = \{e, r_n, \ldots, r_n^{n-1}, s, sr_n, \ldots, sr_n^{n-1}\} = \{r_n^k, sr_n^k \mid k = 0, 1, \ldots, n-1\},$$

with $r_n^n = s^2 = (sr_n)^2 = e$. The element e corresponds to the identity transformation, r_n^j $(j = 1, \ldots, n-1)$ to the counter-clockwise rotation around the Z-axis at an angle of $2\pi j/n$ and s to the reflection with respect to the XZ-plane.

Subgroups of D_n consist of dihedral and cyclic groups whose degree m divides n; i.e. the family of the subgroups of D_n is given by

$$\{D_m^j \mid j = 1, \ldots, n/m;\ m \text{ divides } n\} \quad \text{and} \quad \{C_m \mid m \text{ divides } n\}, \tag{5}$$

where

$$D_m^j = \{r_n^{kn/m}, sr_n^{j-1+kn/m} \mid k = 0, 1, \ldots, m-1\};$$

$$C_m = \{r_n^{kn/m} \mid k = 0, 1, \ldots, m-1\}.$$

Note that $D_m^1 = D_m$, $C_1 = \{e\}$.

These subgroups categorize deformation patterns of regular polygons. Cyclic groups C_m denote rotation-symmetric patterns; the subscript m denotes the number of rotation symmetries, and the group C_1 represents completely asymmetric patterns. Dihedral groups D_m^j indicate line-symmetric patterns; and the subscript m represents the number of axis of line symmetry.

The number of elements of a (sub)group, termed the order, expresses the level of symmetry. The index, which is the ratio of the order of a group to that of a subgroup, stands for the difference of the level of symmetry between these groups. For example, the order of the group D_m is $2m$, denoted as $|D_m| = 2m$; the index of the subgroup D_m in the group D_n is

† For systems with axisymmetry (being equivariant to the continuous group D_{∞}), there appears a group-theoretic double point with D_n-symmetry. It is well known that the resulting bifurcating branches from this point form a sheet, which is obtained merely by rotating a representative bifurcating branch. In this context, our study could also be applicable to the secondary bifurcation structure of a D_{∞}-symmetric system.

‡ In the Schoenflies notation this group is denoted as C_{nv}, whereas D_n means another (isomorphic) group in which s represents the half-rotation around the X-axis. They are isomorphic as abstract groups but are distinguished as subgroups of all isometries.

$$|D_n|/|D_m| = (2n)/(2m) = n/m.$$

4. BIFURCATION BEHAVIOR OF D_n- AND C_n-SYMMETRIC PATHS

In order to categorize bifurcation paths and points of D_n-equivariant systems, we need to introduce the standard (isotypic) decomposition of the real space $X = \mathbf{R}^N$ with respect to D_n (see, e.g., Serre, 1977):

$$X = \begin{cases} X_{n+} \oplus X_{n-} \oplus X_{n/2+} \oplus X_{n/2-} \oplus (\oplus_{k=1}^{n/2-1} X_k), & \text{for } n \text{ even}, \\ X_{n+} \oplus X_{n-} \oplus (\oplus_{k=1}^{(n-1)/2} X_k), & \text{for } n \text{ odd}, \end{cases} \tag{6}$$

where X_{n+}, X_{n-}, $X_{n/2+}$, $X_{n/2-}$ and X_k are mutually orthogonal subspaces of X; k is an integer satisfying $1 \leqslant k < n/2$; and the symbol $\oplus$ expresses the direct sum. Each subspace corresponds to an irreducible representation; X_{n+}, X_{n-}, $X_{n/2+}$ and $X_{n/2-}$ to the irreducible representations of degree one, and $X_k (1 \leqslant k < n/2)$ to those of degree two. In particular, X_{n+} corresponds to the unit representation. Each space is an invariant set under the group action G; for example, $T(g)X_k \subseteq X_k$ for all elements g of D_n.

As shown in Fujii *et al.* (1982), a critical eigenvector at a simple critical point belongs to either the space X_{n+}, X_{n-}, $X_{n/2+}$ or $X_{n/2-}$. On the other hand, two eigenvectors at a group-theoretic double point belong to the subspace X_k for some k.

Each subspace X' of X is associated with a subgroup $g[X']$ of D_n expressing the symmetry of the subspace, i.e.

$$g[X'] = \{g \in D_n | T(g)\underline{x} = \underline{x}, \quad \text{for all } \underline{x} \in X'\}.$$

We have

$$g[X_{n+}] = D_n; \quad g[X_{n-}] = C_n; \quad g[X_{n/2+}] = D_{n/2};$$
$$g[X_{n/2-}] = D_{n/2}^{2} \quad \text{and} \quad g[X_k] = C_m,$$

where m is the greatest common divisor of n and k. The number m is an integer inherent to each double point.

Likewise, the standard decomposition of the real vector space X with respect to C_n reads:

$$X = \begin{cases} \tilde{X}_n \oplus \tilde{X}_{n/2} \oplus (\oplus_{k=1}^{n/2-1} \tilde{X}_k), & \text{for } n \text{ even}, \\ \tilde{X}_n \oplus (\oplus_{k=1}^{(n-1)/2} \tilde{X}_k), & \text{for } n \text{ odd}, \end{cases} \tag{7}$$

where $\tilde{X}_n$, $\tilde{X}_{n/2}$ and $\tilde{X}_k$ are mutually orthogonal subspaces of X. ($\tilde{X}_n$ and $\tilde{X}_{n/2}$ correspond to the irreducible representations of degree one and $\tilde{X}_k$ to those of degree two.) A critical eigenvector at a simple critical point belongs to either $\tilde{X}_n$ or $\tilde{X}_{n/2}$, the symmetry of which is given by

$$g[\tilde{X}_n] = C_n; \quad g[\tilde{X}_{n/2}] = C_{n/2}.$$

A pair of critical eigenvectors at a group-theoretic double point both belong to $\tilde{X}_k$, which is labeled by

$$g[\tilde{X}_k] = C_m,$$

where, as before, m is the greatest common divisor of n and k.

Simple critical points

According to the standard decomposition (6) for D_n, a critical eigenvector for a simple critical point is invariant under either

$$D_n, C_n, D_{n/2} \text{ or } D_{n/2}^2,$$

where a $D_{n/2}$- and a $D_{n/2}^2$-symmetric eigenvector arise only if n is even.

Similarly from eqn (7) for C_n, a critical eigenvector for a simple critical point is invariant under either

$$C_n \text{ or } C_{n/2},$$

where a $C_{n/2}$-symmetric eigenvector arises only if n is even.

In view of the symmetry pattern of its critical eigenvector, a simple critical point on a D_n- (respectively C_n-)symmetric path can be categorized as below.

(1) If the eigenvector is D_n- (respectively C_n-)symmetric, the point is generically a limit point of loading parameter f.

(2) If it is not D_n- (respectively C_n-)symmetric, the point is a simple bifurcation point.

At a simple bifurcation point, two paths branch in the positive and the negative directions of its critical eigenvector. The subgroup that labels the critical eigenvector also labels the symmetry patterns of these paths. The subgroup in this manner corresponds one-to-one to the type of paths.

It follows from eqn (2) that if $\underline{x}$ is a solution for $\underline{H}(f, \underline{x}) = \underline{0}$, then $T(g)\underline{x}$ also satisfies $\underline{H} = \underline{0}$, that is,

$$\underline{H}(f, T(g)\underline{x}) = \underline{0}, \quad \text{for all } g \in G. \tag{8}$$

Hence for a simple bifurcation point, if (f, x) is a set of solutions for a bifurcation path, that for the other path can be known as $(f, T(s)x)$ for a C_n-symmetric bifurcation path and $(f, T(r_n)x)$ for a $D_{n/2}$- or $D_{n/2}^2$-symmetric path. The difference between the symmetry patterns of the two paths depend only on the location of the observer. We henceforth identify such physically equivalent paths and call them an *independent* bifurcation path. Then only one independent path exists at a simple bifurcation point.

We define a bifurcation point to be symmetric if two paths branching in a direction and its opposite direction correspond to the same independent path. Then a simple bifurcation point is necessarily *symmetric*.

Group-theoretic double bifurcation points

Consider a group-theoretic double point of a D_n- or C_n-equivariant system, and denote by C_m the symmetry group associated with the critical eigenvectors at that point. It should be emphasized in the case of a D_n-equivariant system that there exist a finite number of critical eigenvectors with symmetry higher than C_m, i.e. those which are D_m^j-symmetric for some $j = 1, \ldots, n/m$; these eigenvectors turn out to give the directions of bifurcating branches.

Let e_1 and e_2 be two orthonormal real eigenvectors in the kernel space at the group-theoretic double point. In addition, we can choose e_1 to be invariant under D_m^1 for a D_n-(equivariant) system. Then an arbitrary eigenvector e^* in this subspace is expressed as

$$\underline{e}^* = 2w_1 e_1 + 2w_2 e_2.$$

With the use of a complex variable $z = w_1 + iw_2$, which indicates the direction of a bifurcation path in the complex plane, this equation yields

$$\underline{e}^* = (\underline{e}_1 + i\underline{e}_2)z + (\underline{e}_1 - i\underline{e}_2)\bar{z}, \tag{9}$$

where i denotes the imaginary unit, and $\overline{(\cdot)}$ the complex conjugate of the relevant variable. Let $z = r \cdot \exp(i\theta)$. Then we see for a D_n-system that the eigenvector e^* *is* D_m^j-symmetric ($j = 1, \ldots, n/m$) for $2n/m$ angles

$$\theta = \alpha_j = \pi(j-1)m/n, \quad j = 1,\ldots,2n/m. \tag{10}$$

The two-dimensional kernel space can be identified with the space of variables z and $\bar{z}$. It is emphasized that not all of these eigenvectors, in reality, specify the directions of bifurcation paths, but $(z, \bar{z})$ should satisfy two-dimensional bifurcation equations [$M = 2$ in eqn (3)],

$$h_i(z, \bar{z}) = 0, \quad i = 1, 2. \tag{11}$$

Whereas the bifurcation behavior for D_n- and C_n-systems is investigated in Appendix B from bifurcation eqns (11) by imposing the condition of group symmetry (4), the major findings of this investigation are summarized below. It is interesting that the existence of a potential function for the equilibrium eqns (1) [or, the reciprocity of the system (1)] plays a substantial role for the existence of bifurcation paths of a C_n-system. Propositions 2–5 apply both to a D_n-system and a C_n-system.

Proposition 1
(1) For a D_n-system, bifurcation paths always exist irrespective of the presence of a potential function. The D_n-symmetry implies that the potential exists in an asymptotic sense in the vicinity of the double point.
(2) For a C_n-system, bifurcation paths exist if it has a potential function. (The presence of C_n-symmetry alone cannot assure the existence of bifurcation paths.)

Here and henceforth, a C_n-system is assumed to have a potential function so as to insure the existence of bifurcation paths.

Proposition 2
The number of bifurcation paths is $2n/m$, being twice the index n/m. According to Appendix B, we can define these $2n/m$ bifurcation paths as:

$$P_j: \text{path branching toward } \theta = \alpha_j + \beta, \quad j = 1,\ldots,2n/m, \tag{12}$$

where β is non-zero for a C_n-system but vanishes for a D_n-system. Then the path branching in the opposite direction of P_j is expressed as $P_{j+n/m}$.

Proposition 3
The $2n/m$ bifurcation paths $P_j (j = 1,\ldots,2n/m)$ are divided into two independent bifurcation phenomena according to the parity of j, irrespective of degree n and index n/m. Every other path in the θ-direction, therefore, is associated with the same phenomenon.

Proposition 4
(1) The point is symmetric when the index n/m is even and asymmetric when odd.
(2) When the index is odd, a pair of paths P_j and $P_{j+n/m}$, branching in opposite directions, represent two independent physical phenomena, respectively.
(3) When the index is even, the pair of paths correspond to a physically equivalent phenomenon.

Proposition 5
(1) For $n/m = 3$, the loading parameter f decreases toward one independent path but increases toward the other. Both branching paths and the bifurcation point are unstable.
(2) For $n/m = 4$, there are three possibilities for the increase or decrease of the loading parameter f toward the two independent paths. At most one of the independent paths is stable.
(3) For $n/m \geqslant 5$, f increases or decreases at the same time toward the two independent paths. In the former case, one of the paths is stable and the other unstable; the bifurcation point is stable. In the latter case, both paths and the point are unstable.

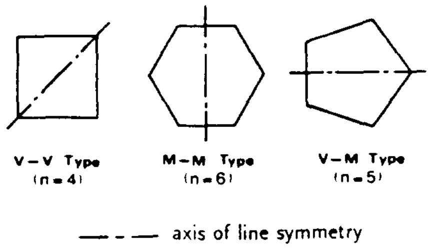

Fig. 4. Three types of axes of line symmetry.

See Fig. B1 in Appendix B for details of this proposition for a D_n-system.

Proposition 6
(1) For a D_n-system, it is only in the directions of D_m^j-symmetric eigenvectors ($j = 1,\dots,n/m$) that bifurcation paths exist.
(2) For a C_n-system, bifurcation paths exist toward C_m-symmetric eigenvectors.

For a D_n-system, the bifurcation paths always have line-symmetric patterns D_m^j ($j = 1,\dots,n/m$), with higher symmetry than the symmetry C_m of the whole kernel space. Symmetry is maximized along the paths, just as maximum or minimum principles govern various kinds of physical phenomena. Such maximization of symmetry does not occur for a C_n-system.

Proposition 7
For a D_n-system with n/m even, D_m^j-symmetric ($j = 1,\dots,n/m$) bifurcation paths are further categorized into two independent physical phenomena according to the parity of j.

This categorization arises from the location of the axis of line symmetry. Figure 4 shows three types of axis for an n-gon: an axis of V–V type connecting two opposite vertices, that of M–M type intersecting two opposite middle points, and that of V–M type connecting a vertex and a middle point. Subgroups D_m^j of D_n with odd degree n possess the axis of V–M type only. Subgroups D_m^j of D_n with even degree n and odd index n/m have both the axis of V–V and M–M type. The axes for D_m^{2j} (respectively D_m^{2j-1}) with even n/m are all of V–V (respectively M–M) type, or all of M–M (respectively V–V) type.

Possible direct branches

Generically possible bifurcation points on a D_n- and C_n-symmetric paths are either a simple critical point or a group-theoretic double one. We offer in Table 1 a complete categorization of these points and their bifurcating branches.

Table 1(a). Direct branches of D_n-symmetric paths.

Symmetry of bifurcation paths	Type of n	Index n/m	Type of bifurcation points	
			Multiplicity	Symmetry
$D_{n/2}$, $D^2_{n/2}$, or C_n	Even	2	Simple	Symmetric
D_m^j ($j = 1,\cdots,n/m$)	Even/Odd	Odd	Double	Asymmetric
D_m^{2j-1} and D_m^{2j} ($j = 1,\cdots,n/2m$)	Even	Even	Double	Symmetric

m divides n and satisfies $m < n/2$

Table 1(b). Direct branches of C_n-symmetric paths.

Symmetry of bifurcation paths	Type of n	Index n/m	Type of bifurcation points	
			Multiplicity	Symmetry
$C_{n/2}$	Even	2	Simple	Symmetric
C_m	Even/Odd	Odd	Double	Asymmetric
	Even	Even	Double	Symmetric

m divides n and satisfies $m < n/2$

The parity of n determines the variety of symmetry patterns of bifurcation paths, which are labeled by subgroups of D_n or C_n. For a D_n-symmetric main path, in fact, the bifurcation paths are either C_n- or D^j_m-symmetric ($j = 1, \ldots, n/m$) for n odd, and either C_n-, $D_{n/2}$-, $D^2_{n/2}$-, D^j_m- ($j = 1, \ldots, n/m$), D^{2j-1}_m- ($j = 1, \ldots, n/2m$) or D^{2j}_m-symmetric ($j = 1, \ldots, n/2m$) for n even. The C_n-, $D_{n/2}$-, and $D^2_{n/2}$-symmetric paths branch at symmetric simple bifurcation points; D^j_m, D^{2j-1}_m and D^{2j}_m-symmetric ones at group-theoretic double points.

It is to be noted that C_m-symmetric paths with $m < n$ cannot branch directly from D_n-symmetric paths so that not all subgroups of D_n are feasible as the symmetry group of direct branches of D_n. The direct branches have line-symmetric bifurcation patterns, except for C_n-symmetric ones branching at a simple point.

For a C_n-symmetric path, the bifurcation paths are C_m-symmetric with $m < n$. The $C_{n/2}$-symmetric path branches at a symmetric simple bifurcation point for n even, and the C_m-symmetric one with $m < n/2$ at a group-theoretic double point.

Implementation in numerical analysis

Based on the results presented in the previous subsections, the following procedure is suggested for the numerical analysis of a D_n- (respectively C_n-) equivariant bifurcation system.

(1) Obtain a D_n- (respectively C_n-)equivariant path. At the same time, critical points on the path and critical eigenvectors of the points are obtained through the eigenvalue analysis of the tangent-stiffness matrix. (From an engineering standpoint, one often has to obtain only the first critical point.)

(2) Determine the type of points with reference to Table 1 by investigating the rank deficiency of the matrix and the symmetry pattern of the critical eigenvectors.

(3) Obtain an independent bifurcation path at a simple bifurcation point and two independent paths at a group-theoretic double point.

The solution set (f, x) of a path branching at a simple bifurcation point is to be found in the direction of the critical eigenvector, whereas that for the other path can be known automatically as $(f, T(g)x)$ for some $g \in G$.

For a double point on a D_n-symmetric path, it may be difficult to determine the isotypic component X_k to which the two critical eigenvectors belong. Instead of this, the symmetry group $g[X_k]$ of X_k is to be obtained to determine the index n/m. If the index is odd, the bifurcation point is asymmetric, and hence two independent bifurcation paths are to be found in the positive and the negative directions of a D^j_m-symmetric eigenvector for some j. If it is even, the bifurcation point is symmetric, and two independent paths can be found in the directions of D^{2j-1}_m- and of D^{2j}_m-symmetric eigenvectors for some j.

For a double point on a C_n-symmetric path, the directions of two independent paths cannot be known through the symmetry pattern of the critical eigenvectors. These paths, therefore, must be found on a trial basis unless bifurcation equations are derived at a considerable cost.

5. BIFURCATION HIERARCHY

Direct bifurcation paths of D_n- and C_n-symmetric paths have been identified in the previous section. These bifurcation paths can undergo further progressive symmetry-breaking bifurcation until reaching the completely asymmetric pattern C_1. Repeated bifurcations make up a hierarchy among bifurcation paths. Since the symmetry group of a bifurcation path is a subgroup of the group of the main path, we may associate a chain of nested subgroups of D_n with a process of repeated bifurcation. Sub-branches of the paths labeled by D^j_m ($j = 1, \ldots, n/m$) or C_m can be analyzed by the recursive use of Table 1. In the analysis of bifurcation hierarchy, the procedure presented in the previous section must be employed repeatedly.

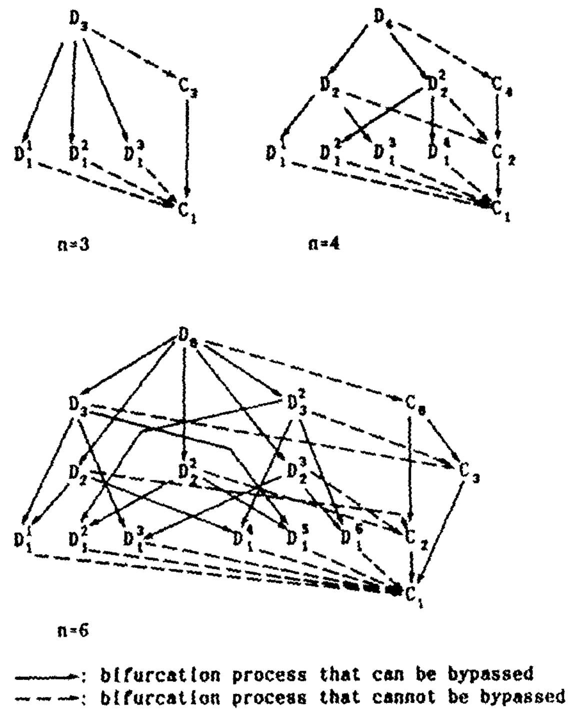

Fig. 5. Hierarchy of bifurcation paths of D_n.

For example, Fig. 5 shows the hierarchies for D_3, D_4 and D_6. Bifurcation progresses in the arrowed directions. During this process, the solid arrows can be bypassed but not the dotted arrows. A bifurcation process $D_4 \rightarrow D_2 \rightarrow D_1^1$, made up of only solid arrows, denotes that a D_1^1-symmetric path can branch directly from a D_4-symmetric one, bypassing a D_2-symmetric one. By contrast, $D_4 \dashrightarrow C_4 \rightarrow C_2$ shows that a C_4-symmetric path can branch from a D_4-symmetric one and a C_2-symmetric one cannot, since the dashed arrow may not be bypassed. A similar diagram has been devised by Dellnitz and Werner (1989).

We may identify two major hierarchies: a hierarchy among line-symmetric paths D_n and D_m^j ($j = 1, \ldots, n/m$) and that among rotation-symmetric ones C_n and C_m. All line-symmetric paths are connected by solid arrows, and so are all rotation-symmetric ones. A line-symmetric path and a rotation-symmetric path with the same degree, in contrast, are connected by a dashed arrow.

The $C_{n/2}$- and the C_m-symmetric bifurcation paths can arise only as a consequence of repeated symmetry-breaking bifurcation; a $C_{n/2}$-symmetric path, for example, can branch only from either a C_n-, $D_{n/2}$- or $D_{n/2}^2$-symmetric path and not directly from a D_n-symmetric path. It may be noted, however, that each subgroup of D_n given in eqn (5) is potentially reachable as a bifurcation path type although its actual existence in a particular bifurcation behavior depends on the numerical properties of the problem in question.

6. NUMERICAL EXAMPLES

The bifurcation behavior of the elastic truss domes of Fig. 1, all subjected to symmetric loadings, is described here by means of the group-theoretic method. All members of these domes have the same material and sectional properties, thus realizing symmetric stiffness distributions.

A truss dome is equivariant to a group if its geometrical configuration, stiffness distribution and loading pattern are all preserved by the transformation by each element of the group. In the discussion of equivariance of domes, their detailed aspects, such as heights, radii and number of degrees of freedom, are immaterial. Instead of these, line and rotation symmetries play a primary role. Hence the regular-triangular dome of Fig. 1(a),

Table 2. Branches of D_6-equivariant systems.

Symmetry of main paths	Symmetry of bifurcation paths	Index n/m	Type of bifurcation points	
			Multiplicity	Symmetry
D_6	D_3, D_3^2 or C_6	2	Simple	Symmetric
	D_2^j	3	Double	Asymmetric
	D_1^{2j-1} and D_1^{2j}	6	Double	Symmetric
D_3	C_3	2	Simple	Symmetric
	D_1^{2j-1}	3	Double	Asymmetric
D_3^2	C_3	2	Simple	Symmetric
	D_1^{2j}	3	Double	Asymmetric
C_6	C_3	2	Simple	Symmetric
	C_2	3	Double	Asymmetric
D_2^j	D_1^{2j-1}, D_1^{2j}, or C_2	2	Simple	Symmetric
C_3	C_1	3	Double	Asymmetric
D_1^{2j-1}, D_1^{2j}, or C_2	C_1	2	Simple	Symmetric

$j = 1,2,3$

the regular-hexagonal dome of Fig. 1(b) and the rotation-symmetric dome of Fig. 1(c) are equivariant to D_3, D_6 and C_6, respectively. Table 2 shows a categorization of bifurcation paths and points of D_6-equivariant systems obtained from Table 1. Bifurcation rules for D_3 and C_6, which are subgroups of D_6, can be seen in a part of those for D_6 in Table 2.

D_3-equivariant dome

The bifurcation behavior of the D_3-equivariant truss dome, which has been introduced in Section 2, is described here by means of the group-theoretic bifurcation theory presented in the previous sections.

The critical eigenvectors at the unstable double bifurcation point a on the D_3-symmetric path are labeled by either D_1^j ($j = 1,2,3$) or C_1. The index of D_1^j in D_3 is $|D_3|/|D_1^j| = 3$ ($j = 1,2,3$), and its parity is odd. According to Propositions 2, 4, 5 and 6, this is an asymmetric point with six D_1^j-symmetric paths P_j and P_{j+3} ($j = 1,2,3$), as we have already seen in Fig. 3(a). The paths P_1, P_3 and P_5 (respectively P_2, P_4 and P_6) correspond to an independent path, in agreement with Proposition 4.

The bifurcation points c and d, where C_1-symmetric paths branch from the D_1^j-symmetric ($j = 1,2,3$) paths, are simple and symmetric points. An independent path branches at these simple points, whereas two independent paths branch at double points a and b, in accordance with Proposition 3.

All these bifurcation phenomena obey the rules in Table 2 and follow the hierarchy in Fig. 5. The C_3-symmetric paths, which are theoretically feasible, are absent in these equilibrium paths. The tasks involved in finding bifurcation paths, especially at double points, have been reduced greatly owing to the procedure for obtaining bifurcation paths presented in the previous section.

D_6-equivariant dome

The bifurcation patterns of the D_6-equivariant spherical diamond shell of Fig. 1(b) are expressed by the subgroups of D_6; namely, D_6, D_3, D_3^2, D_2^j ($j = 1,2,3$), D_1^{2j-1} ($j = 1,2,3$), D_1^{2j} ($j = 1,2,3$), C_6, C_3, C_2 and C_1. Figure 6 shows these patterns in a plane view. The nodes with the same vertical and radial displacements are denoted by the same symbol, such as (●), (○), ..., (□). The nodes on the axis of line symmetry do not rotate but stay on the axis. The nodes elsewhere rotate in such a manner that they satis^fy the required line and the rotation symmetries.

We computed equilibrium paths for the dome under D_6-symmetric vertical loadings applied in such a manner that the vertical component of the loading pattern vector is equal to 0.5 for the crown node and to unity for other free nodes. Figure 7 shows a schematic view of these paths. The numbers at the bifurcation points stand for the index. For example, the index is equal to three for asymmetric double points c and f, to six for a symmetric double

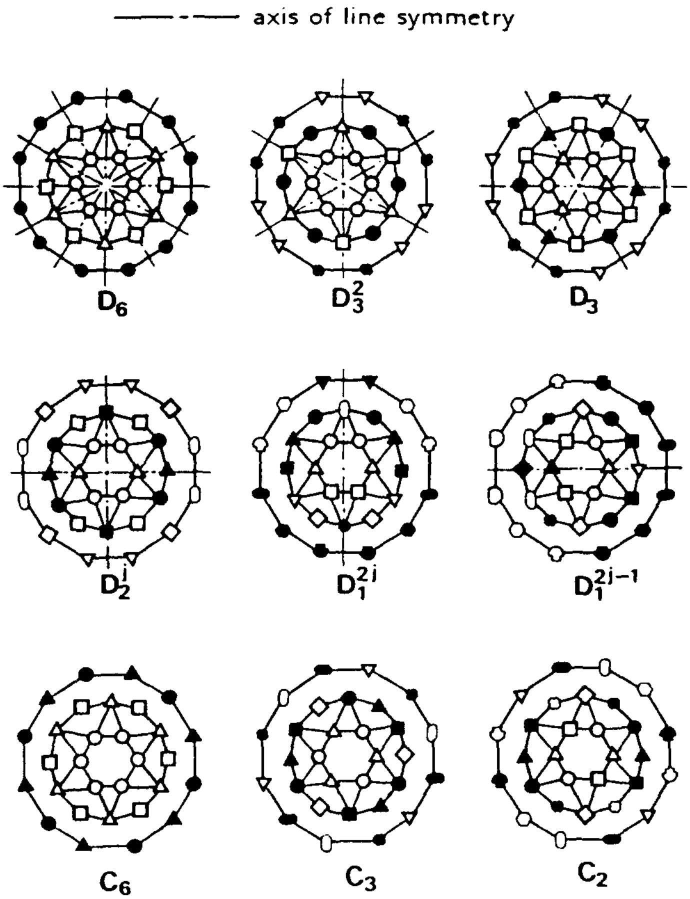

Fig. 6. Schematic plane views of bifurcation patterns of the spherical diamond shell (D_6-equivariant).

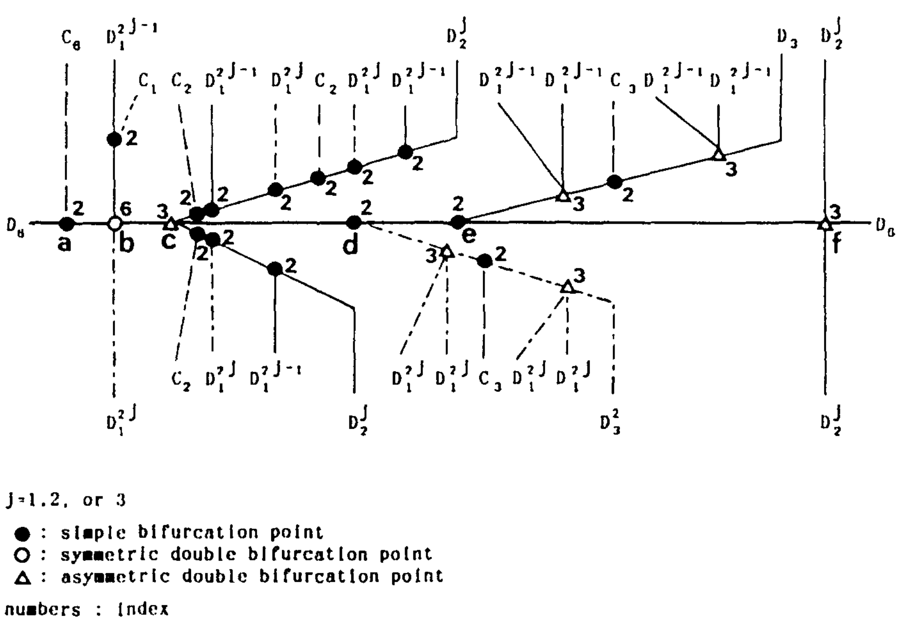

Fig. 7. Schematic view of bifurcation hierarchy of the spherical diamond shell (D_6-equivariant).

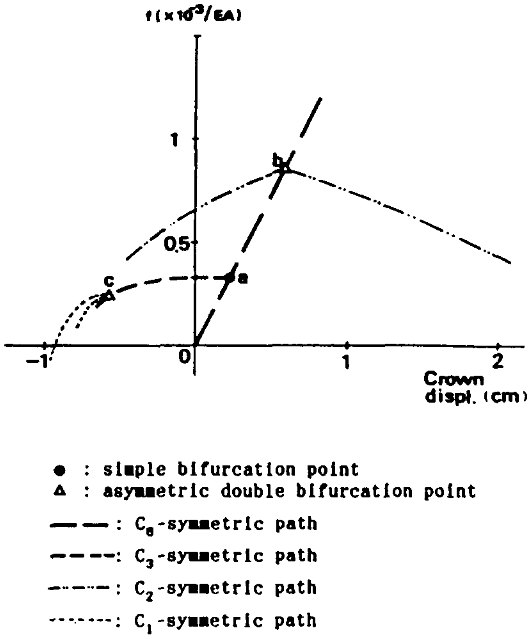

Fig. 8. Equilibrium paths for the Schwedler dome (C_6-equivariant).

point b, and to two for symmetric simple bifurcation points a, d and e. An independent path branches at simple points and two independent ones do at double points; the bifurcation points are symmetric for even indices and asymmetric for odd ones. All these bifurcation phenomena obey the rules in Table 2 and those propositions for double points. In addition, all subgroups of D_6 are observed as the labels for the symmetry of paths.

C_6-equivariant dome

Bifurcation patterns of the C_6-equivariant Schwedler dome of Fig. 1(c) are expressed by the subgroups of C_6, i.e. C_6, C_3, C_2 and C_1. Figure 8 shows solution paths of the dome computed for symmetric vertical loadings whose vertical components equal 0.5 for the inner hexagonal nodes and unity for the remaining nodes. From the C_6-symmetric fundamental path, C_3-symmetric paths branch at a symmetric simple bifurcation point a, and C_2-symmetric paths branch at an asymmetric double point b with an odd index $|C_6|/|C_2| = 3$. C_1-symmetric paths branch from the C_3-symmetric paths at an asymmetric double point c with an odd index $|C_3|/|C_1| = 3$. Again, those propositions have been satisfied.

7. CONCLUSIONS

In this paper, we have organized and developed the results of group-theoretic bifurcation theory and of a heuristic case study to present a method to describe bifurcation hierarchy of symmetric (structural) systems equivariant to D_n or C_n. With the aid of bifurcation equations, we have noted that the nature of group-theoretic double bifurcation points depends entirely on a single variable, namely the index, expressing the degradation of symmetry at the points. In addition, it is remarked that the bifurcation paths branch at a group-theoretic double point on a C_n-equivariant system if it has a potential function.

With the use of this method, highly complex bifurcation phenomena of D_n- and C_n-equivariant systems can be understood as a hierarchy of symmetry-breaking bifurcation. *A priori* knowledge of bifurcation hierarchy permits one to analyze the bifurcation phenomena in a systematic manner. In particular, the complete set of bifurcation paths branching at group-theoretic double points can be arrived at with reduced work.

Acknowledgements—The authors are grateful for invaluable comments offered by Professor Martin Golubitsky. The comments of the anonymous referees were helpful in revision. Shogo Matsushita and Shosaku Wada were of great assistance in the numerical analysis. This work was carried out while the second author stayed at the Institute of Operations Research, University of Bonn, supported by the Alexander von Humboldt Foundation.

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APPENDIX A: A REMARK ON BIFURCATION EQUATIONS OF GRADIENT SYSTEMS

A system of nonlinear eqns (1) is called a gradient system if

$$\underline{H} = [H_i] = \text{grad } U \tag{A1}$$

for some scalar (potential) function U. Note that eqns (A1) imply the reciprocity of eqns (1), i.e. the symmetry of the Jacobian matrix of eqns (1):

$$\partial H_i/\partial x_j = \partial H_j/\partial x_i. \tag{A2}$$

Since the reciprocity (A2) implies eqns (A1) under some regularity conditions by the Poincaré lemma, the concepts of "gradient systems" and "reciprocal systems" are in fact equivalent.

The observation here is the following theorem.

Theorem A. If the full eqns (1) *are a gradient system, so are the reduced bifurcation eqns* (3) (*by an appropriate choice of coordinates*). *Or in other words, the reduced bifurcation eqns* (3) *inherit the reciprocity of the full eqns* (1).

This principle is independent of, and may be compared to, the well-known important principle (4) that the reduced bifurcation equations inherit the group symmetry of the full equations. Thus we have two independent general properties which are preserved under the Lyapunov–Schmidt procedure. Note also that it is straightforward to extend this observation to infinite-dimensional case, using the usual framework of a Fredholm operator on a Hilbert space.

The proof of the above theorem is not difficult, as follows. By an appropriate change of the coordinates, we may assume, using the notation of Section 3, that $\underline{x} = (\underline{w}, \underline{v})$ (where $\underline{w} \in \mathbf{R}^M, \underline{v} \in \mathbf{R}^{N-M}$) and $\underline{v}$ can be expressed as $\underline{v} = \underline{v}(\tilde{f}, \underline{w})$ by the implicit function theorem and that the functions $h_i(\tilde{f}, \underline{w})$ of eqns (3) are given by

$$h_i(\tilde{f}, \underline{w}) = H_i(\tilde{f}, \underline{w}, \underline{v}(\tilde{f}, \underline{w})).$$

Then the Jacobian matrix $J' = (\partial h_i/\partial w_j)$ for the reduced equations is given by

$$J' = J_{11} - J_{12}J_{22}^{-1}J_{21}$$

in terms of the submatrices $J_{ij}(i, j = 1, 2)$ of the Jacobian matrix J of the full eqns (1), i.e. $J = (J_{ij}|i, j = 1, 2)$ and $J_{11} = (\partial H_i/\partial w_j|i, j = 1, \ldots, M)$, $J_{12} = (\partial H_i/\partial w_j|i = 1, \ldots, M; j = M+1, \ldots, N)$, etc. The symmetry of J implies that of J'. This completes the proof.

Consider a double critical point (i.e. $M = 2$). By choosing $(z, \bar{z})$ in eqn (9) as the coordinates of the kernel space, we put

$$F(\hat{f}, z, \bar{z}) = h_1(\hat{f}, w_1, w_2) + ih_2(\hat{f}, w_1, w_2),$$

where $\hat{f}$ is an increment of f from its value at the double point. Let us assume that F can be expanded as

$$F(\hat{f}, z, \bar{z}) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq}(\hat{f}) z^p \bar{z}^q.$$

The reciprocity can be expressed in a compact form as follows.

Theorem B. At a double point, the reduced eqns (3) *are reciprocal if and only if $\partial F/\partial z$ is real. This condition is equivalent to*

$$(p+1)A_{p+1,q}(\hat{f}) = (q+1)\overline{A_{q+1,p}(\hat{f})}, \quad p, q = 0, 1, 2, \ldots. \tag{A3}$$

Note that the above statement is independent of whether the double point is parametric or group-theoretic. An important consequence of eqn (A3) is that

$$A_{p+1,p}(\hat{f}) \text{ is real for } p = 0, 1, 2, \ldots, \tag{A4}$$

which plays a key role in the analysis of a double point of a C_n-equivariant system; see Appendix B.

APPENDIX B: GROUP-THEORETIC DOUBLE BIFURCATION POINTS OF D_n- AND C_n-SYSTEM

The solutions of the two-dimensional bifurcation eqns (11) of a group-theoretic double point of a D_n-equivariant system have been obtained in mathematics (see, e.g., Sattinger, 1979, 1980; Golubitsky *et al.*, 1988). In order to make these solutions accessible for structural engineers, they are presented here by means of elementary calculations.

We consider a double critical point (i.e. $M = 2$) and identify the kernel space with the space of $(z, \bar{z})$ through eqn (9), i.e. $z = w_1 + iw_2$, $\bar{z} = w_1 - iw_2$. If we put

$$F(\hat{f}, z, \bar{z}) = h_1(\hat{f}, w_1, w_2) + ih_2(\hat{f}, w_1, w_2),$$

where $\hat{f}$ is an increment of f from its value at the double point ($\hat{f} = 0$ at the point), we see that eqns (11) are equivalent to a complex equation

$$F(\hat{f}, z, \bar{z}) = 0 \tag{B1}$$

since h_1 and h_2 are real. Suppose we can expand F as

$$F(\hat{f}, z, \bar{z}) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq}(\hat{f}) z^p \bar{z}^q. \tag{B2}$$

(As is usual with local bifurcation analysis, only finitely many terms are important.) Assuming that $(\hat{f}, w_1, w_2) = (0, 0, 0)$ corresponds to the double critical point, we have

$$A_{00}(0) = A_{10}(0) = A_{01}(0) = 0. \tag{B3}$$

As we will see below, A_{pq} must follow certain conditions if the system has a potential function (as in Appendix A) or a D_n- (or C_n-)symmetry (as shown below).

D_n-equivariant system

The D_n-equivariance (4) at a group-theoretic point is expressed as follows. Let C_m be the group of symmetry of the kernel space of the double point. By the choice of e_1, D_n acts on $(z, \bar{z})$ via

$$r_n \cdot z = \omega z, \quad r_n \cdot \bar{z} = \bar{\omega}\bar{z}, \quad s \cdot z = \bar{z}, \quad s \cdot \bar{z} = z,$$

where

$$\omega = \exp(i2\pi m/n).$$

Note that $n/m \geqslant 3$. Then the D_n-equivariance (4) is equivalent to

$$\omega F(\hat{f}, z, \bar{z}) = F(\hat{f}, \omega z, \bar{\omega}\bar{z}). \tag{B4}$$

$$\overline{F(\hat{f}, z, \bar{z})} = F(\hat{f}, \bar{z}, z). \tag{B5}$$

From eqn (B5) we see that

$$A_{pq}(\hat{f}) \text{ is real for } p, q = 0, 1, \ldots . \tag{B6}$$

Substitution of eqn (B2) in (B4) yields

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq}(\hat{f}) z^p \bar{z}^q [\exp\{i2\pi(p-q-1)m/n\} - 1] = 0.$$

Hence,

$$A_{pq}(\hat{f}) = 0 \quad \text{unless } p-q-1 = kn', \quad k = 0, \pm 1, \pm 2, \ldots, \tag{B7}$$

where $n' = n/m$.

Equations (B6) and (B7) are the conditions for D_n-symmetry. Under condition (B7), eqn (B2) is rewritten as

$$F(\hat{f}, z, \bar{z}) = \sum_{q=0}^{\infty} A_{q+1,q}(\hat{f}) z^{q+1} \bar{z}^q + \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} [A_{q+1+kn',q}(\hat{f}) z^{q+1+kn'} \bar{z}^q + A_{q,q-1+kn'}(\hat{f}) z^q \bar{z}^{q-1+kn'}]. \tag{B8}$$

We define the leading part of this equation as:

$$F_0(\hat{f}, z, \bar{z}) = \sum_{0 \le q \le n'/2-1} A_{q+1,q}(\hat{f}) z^{q+1} \bar{z}^q + A_{0,n'-1}(\hat{f}) \bar{z}^{n'-1}. \tag{B9}$$

A key observation here is that the leading part in eqn (B9) satisfies the condition of reciprocity eqn (A3). This shows that D_n-symmetry implies the existence of a potential function in the asymptotic sense. Put

$$\tilde{h}_1(w_1, w_2) = \mathcal{R}(F_0(\hat{f}, z, \bar{z})),$$
$$\tilde{h}_2(w_1, w_2) = \mathcal{I}(F_0(\hat{f}, z, \bar{z})),$$

where $\mathcal{R}(\cdot)$ and $\mathcal{I}(\cdot)$ mean the real and imaginary parts of a complex number. Then we have

$$h_1(w_1, w_2) \approx \tilde{h}_1(w_1, w_2) = \partial U/\partial w_1,$$
$$h_2(w_1, w_2) \approx \tilde{h}_2(w_1, w_2) = \partial U/\partial w_2$$

for some "asymptotic potential" function $U(w_1, w_2)$. This function is evaluated to

$$U(w_1, w_2) = \int_0^{w_1} \tilde{h}_1(w_1, w_2)\, dw_1 + \int_0^{w_2} \tilde{h}_2(0, w_2)\, dw_2$$
$$= \sum_{0 \le q \le n'/2-1} \frac{A_{q+1,q}(\hat{f})}{2(q+1)} (w_1^2 + w_2^2)^{q+1} + A_{0,n'-1}(\hat{f}) \mathcal{R}((w_1 + iw_2)^{n'})/n'.$$

In polar coordinates $z = r \cdot \exp(i\theta)$, this function becomes

$$U(w_1, w_2) \equiv \tilde{U}(r, \theta) = \sum_{0 \le q \le n'/2-1} \frac{A_{q+1,q}(\hat{f})}{2(q+1)} r^{2(q+1)} + A_{0,n'-1}(\hat{f}) r^{n'} \cos(n'\theta)/n'$$
$$\approx A'_{10}(0)\hat{f} r^2/2 + \sum_{1 \le q \le n'/2-1} \frac{A_{q+1,q}(\hat{f})}{2(q+1)} r^{2(q+1)} + A_{0,n'-1}(\hat{f}) r^{n'} \cos(n'\theta)/n' \tag{B10}$$

by eqn (B3), where A'_{10} denotes the derivative of A_{10} with respect to $\hat{f}$.

The equilibrium eqn (B1) has the trivial solution $z = 0$, corresponding to the D_n-symmetric main path. Their non-trivial solution is determined from $F/z = 0$. Putting

$$\tilde{F}(\hat{f}, r, \theta) = F(\hat{f}, r\exp(i\theta), r\exp(-i\theta))/r\exp(i\theta)$$

and using polar coordinates and eqn (B7), we have

$$\mathcal{R}(\tilde{F}) = \sum_{q=0}^{\infty} A_{q+1,q}(\hat{f}) r^{2q} + \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} [A_{q+1+kn',q}(\hat{f}) r^{2q+kn'} + A_{q,q-1+kn'}(\hat{f}) r^{2(q-1)+kn'}] \cos(kn'\theta),$$

$$\mathcal{I}(\tilde{F}) = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} [A_{q+1+kn',q}(\hat{f}) r^{2q+kn'} - A_{q,q-1+kn'}(\hat{f}) r^{2(q-1)+kn'}] \sin(kn'\theta). \tag{B11}$$

The non-trivial solution is to satisfy equation $\mathcal{I}(\tilde{F}) = 0$ and hence $\sin(n'\theta) = 0$. Therefore,

$$\theta = \alpha_j = \pi(j-1)m/n, \quad j = 1, \ldots, 2n/m$$

are necessary for the existence of non-trivial solutions. These angles are associated with the directions of D'_m-symmetric eigenvectors ($j = 1, \ldots, n/m$) [see eqn (10)].

If we put

$$\tilde{F}_i(\hat{f}, r) = \tilde{F}(\hat{f}, r, \alpha_i), \quad i = 1, 2,$$

we see the symmetry among the paths from eqn (B11), that is,

$$\tilde{F}_i(\hat{f}, r) = \tilde{F}(\hat{f}, r, \alpha_{2(j-1)+i}), \quad j = 1, \ldots, n/m, \quad i = 1, 2. \tag{B12}$$

It can be proven by the implicit function theorem that, for each i, $\tilde{F}_i(\hat{f}, r) = 0$ can be solved generically for $\hat{f}$ as

$$\hat{f} = f_i(r) \tag{B13}$$

in the neighborhood of the double point $(\hat{f}, r) = (0, 0)$. Therefore the number of bifurcation paths equals $2n/m$, which is twice the index $|D_n|/|D'_m|$.

The above argument [see eqn (B12)] shows the existence of two distinct sets of bifurcation paths denoted by $f_1(r)$ and $f_2(r)$. Hence the $2n/m$ bifurcation paths are divided into two physically independent paths. Every other bifurcation path in the θ-direction is associated with an independent path.

We denote by P_j the path branching in the direction of $\theta = \alpha_j$ ($j = 1, \ldots, 2n/m$). Paths P_{2j-1} ($j = 1, \ldots, n/m$) are described by $f_1(r)$ and are D_m^{2j-1}-symmetric; paths P_{2j} by $f_2(r)$ and are D_m^{2j}-symmetric.

A double bifurcation point is symmetric if a pair of paths P_j and $P_{j+n/m}$, which branch in opposite directions $\theta = \alpha_j$ and $\alpha_j + \pi$, correspond to the same function f_i ($i = 1$ or 2); it is asymmetric otherwise.

For n/m odd, the pair of paths are denoted by different functions so that the bifurcation point is asymmetric. Two different paths $f_1(r)$ and $f_2(r)$ are connected at the bifurcation point to form a continuous D'_m-symmetric path ($j = 1, \ldots, n/m$), which is asymmetric with respect to the main path. For n/m even, the pair of paths are symmetric with respect to the main path, and hence the bifurcation point is symmetric.

Direct calculations reveal the following asymptotic forms of $f_i(r)$, $i = 1, 2$, when $|r|$ is small. For $n/m = 3$,

$$f_i(r) \approx (-1)^i \frac{A_{02}(0)}{A'_{10}(0)} r, \quad i = 1, 2.$$

Since $f_1(r)$ and $f_2(r)$ have opposite signs, f is always reduced toward one independent path but increases toward the other. For $n/m = 4$, eqn (B11) yields

$$f_i(r) \approx -\frac{1}{A'_{10}(0)} \{A_{21}(0) + (-1)^{i-1} A_{03}(0)\} r^2, \quad i = 1, 2.$$

The coefficients $A_{21}(0) \pm A_{03}(0)$ determine the increase or decrease of f. For $n/m \geq 5$,

$$f_i(r) \approx -\frac{A_{21}(0)}{A'_{10}(0)} r^2 + g(r) + (-1)^i \frac{A_{0,n/m-1}(0)}{A'_{10}(0)} r^{n/m-2} + O(r^{n/m}), \quad i = 1, 2,$$

where $g(r) = O(r^4)$ is independent of i. Thus f increases or decreases simultaneously for all bifurcation paths according to whether $A_{21}(0)/A'_{10}(0)$ is negative or positive.

The stability of the bifurcation point and branches are now considered. Putting $\hat{f}$ to zero in the asymptotic potential $\tilde{U}$ in (B10), we obtain

$$\tilde{U}(r, \theta) \approx \begin{cases} A_{02}(0) \cos(3\theta) r^3/3 & \text{if } n/m = 3, \\ [A_{21}(0) + A_{03}(0) \cos(4\theta)] r^4/4 & \text{if } n/m = 4, \\ A_{21}(0) r^4/4 & \text{if } n/m \geq 5, \end{cases}$$

in the neighborhood of $r = 0$. Remember that the bifurcation point is stable if $\tilde{U}(r, \theta)$ is minimized at the point. The point is a saddle point of $\tilde{U}$ for $n/m = 3$, and therefore is unstable. For $n/m = 4$, it is stable for $A_{21}(0) \pm A_{03}(0) > 0$ but otherwise it is unstable. For $n/m \geq 5$, it is stable if $A_{21}(0)$ is positive but unstable if negative.

Next we consider the stability of branches through linearizations. Let $J' = (\partial H_i/\partial w_j | i, j = 1, 2)$ be the (asymptotically symmetric) Jacobian matrix of the reduced equations and recall that an equilibrium state is stable if J' has two positive eigenvalues, and is unstable if it has at least one negative eigenvalue. Since J' is two-dimensional, it has two positive eigenvalues if and only if both

$$\mathrm{trace}(J') > 0 \quad \text{and} \quad \det(J') > 0,$$

where trace $(\cdot)$ and det $(\cdot)$ is the trace and the determinant of a matrix, respectively. From eqn (A1) we see

$$\mathrm{trace}(J') = 2\mathcal{R}(\partial F/\partial z), \quad \det(J') = |\partial F/\partial z|^2 - |\partial F/\partial \bar{z}|^2.$$

On the bifurcation branch with $\theta = \alpha_i$, we obtain the following expressions when $|r|$ is small:

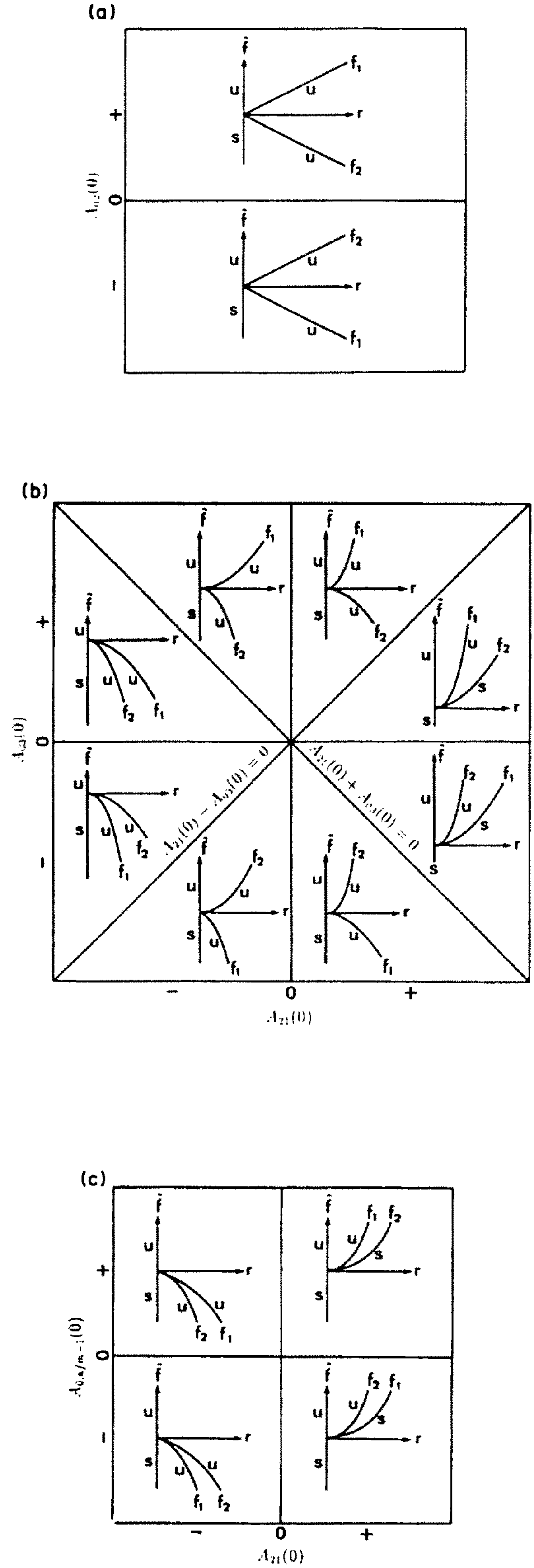

Fig. B1. Categorization of local behavior at the group-theoretic double bifurcation point for D_n: (a) $n/m = 3$; (b) $n/m = 4$; (c) $n/m \geqslant 5$; s: stable path; u: unstable path.

$$\tfrac{1}{2}\,\mathrm{trace}\,(J') \approx \begin{cases} (-1)^l A_{02}(0)r & \text{if } n/m = 3, \\ [A_{21}(0)+(-1)^{l-1}A_{03}(0)]r^2 & \text{if } n/m = 4, \\ A_{21}(0)r^2 & \text{if } n/m \geqslant 5; \end{cases}$$

$$\det(J') \approx \begin{cases} -3A_{02}(0)^2 r^2 & \text{if } n/m = 3, \\ 8A_{03}(0)[(-1)^l A_{21}(0)+A_{03}(0)]r^4 & \text{if } n/m = 4, \\ (-1)^l(2n/m)A_{0,n/m-1}(0)A_{21}(0)r^{n/m} & \text{if } n/m \geqslant 5. \end{cases}$$

Note also that $A'_{10}(0) < 0$ since the trivial solution $r = 0$ is stable for $\tilde{f} < 0$.

From these calculations, we can see the stability of the double point. Figure B1 categorizes the asymptotic behavior of a non-degenerate group-theoretic double bifurcation point for D_n, for which the coefficients $A_{02}(0)$, $A_{03}(0)$, $A_{21}(0) \pm A_{03}(0)$ and $A_{0,n/m-1}(0)$ are assumed to be distinct from zero.

For $n/m = 3$, the bifurcation paths in general are all unstable. For $n/m = 4$, all branches are unstable if $A_{21}(0)-A_{03}(0)$ is negative. If it is positive, the branches P_{2l-1} are unstable and P_{2l} are stable for positive $A_{03}(0)$, and vice versa for negative $A_{03}(0)$. For $n/m \geqslant 5$, all branches are unstable if $A_{21}(0)$ is negative. If it is positive, the branches P_{2l-1} are unstable and P_{2l} are stable for positive $A_{0,n/m-1}(0)$, and vice versa for negative $A_{0,n/m-1}(0)$.

C_n-equivariant system

For a C_n-equivariant system, we have eqn (B7) only and not eqn (B6). Hence $F(\tilde{f}, z, \bar{z})$ is written as eqn (B8) but with complex coefficients $A_{pq}(\tilde{f})$. [Note that $A_{pq}(\tilde{f})$ are real for a D_n-equivariant system due to the reflection symmetry (B5).] Since $A_{q+1,q}(\tilde{f})$ $(q = 0, 1, \ldots)$ are complex in general in eqn (B8), the equation $\tilde{F}(\tilde{f}, z, \bar{z}) = 0$ has no solution. That is, $F(\tilde{f}, z, \bar{z}) = 0$ has the trivial solution alone.

However, if the system has a potential, the additional condition (A4) is satisfied. Then it can be proven by an elementary argument based on the implicit function theorem that $F(\tilde{f}, z, \bar{z}) = 0$ has bifurcating branches in the direction of

$$\theta = \alpha_j + \beta, \quad j = 1, \ldots, 2n/m, \tag{B14}$$

where α_j is defined in eqn (10) and $\beta = \arg(A_{0,n/m-1}(0))$, where $\arg(\cdot)$ is the argument of a complex variable. The presence of this angle β is the only difference between D_n-systems and C_n-systems, though the values of β cannot be known through the consideration of symmetry. Unless higher-order terms are referred to, the determination of β demands a series of preliminary trial-and-error analyses.

Since $A_{pq}(\tilde{f})$ can be complex for a C_n-system, the results obtained for a D_n-system need some modifications. The asymptotic form of $\tilde{f}_i(r)$ $(i = 1, 2)$ of eqn (B13) is given by

$$\tilde{f}_i(r) \approx \begin{cases} \dfrac{(-1)^i |A_{02}(0)|}{A'_{10}(0)} r & \text{if } n/m = 3, \\ -\dfrac{1}{A'_{10}(0)}[A_{21}(0)+(-1)^{i-1}|A_{03}(0)|]r^2 & \text{if } n/m = 4, \\ -\dfrac{A_{21}(0)}{A'_{10}(0)} r^2 & \text{if } n/m \geqslant 5. \end{cases}$$

The stability of the bifurcating paths with $\theta = \alpha_j + \beta$ is determined from the following expressions:

$$\tfrac{1}{2}\,\mathrm{trace}\,(J') \approx \begin{cases} (-1)^l \mathscr{A}(A_{02}(0))r & \text{if } n/m = 3, \\ [A_{21}(0)+(-1)^{l-1}\mathscr{A}(A_{03}(0))]r^2 & \text{if } n/m = 4, \\ A_{21}(0)r^2 & \text{if } n/m \geqslant 5; \end{cases}$$

$$\det(J') \approx \begin{cases} -3|A_{02}(0)|^2 r^2 & \text{if } n/m = 3, \\ 8[(-1)^l \mathscr{A}(A_{03}(0))A_{21}(0)-|A_{03}(0)|^2]r^4 & \text{if } n/m = 4, \\ (-1)^l(2n/m)\mathscr{A}(A_{0,n/m-1}(0))A_{21}(0)r^{n/m} & \text{if } n/m \geqslant 5. \end{cases}$$